Ch.3. Traditional Methods - Part 1

Linear Programming

- LP Problem:

$$\max a_{01}x_1 + \cdots + a_{0n}x_n$$

$$\begin{aligned} s.t.\ \ & a_{i1}x_1 + \cdots a_{in}x_n \le b_i \ \ (i = 1,\cdots,m_1) \\ & a_{j1}x_1 + \cdots a_{jn}x_n \le b_j \ \ (j = m_1+1,\cdots,m_1+m_2) \\ & a_{k1}x_1 + \cdots a_{kn}x_n \le b_k \ \ (k = m_1+m_2+1,\cdots,M) \end{aligned}$$

$$x_1 \ge 0, \cdots, x_n \ge 0$$

LP problem → CP (convex programming) problem (ex) Production scheduling problem

- Profit

  chair : \$20 / unit

  table : \$30 / unit

- Requirements

	Wood (unit)	Labor (hour)
Chair	1	3
Table	6	1
Total available	288	99

- Problem

  determines the number of chairs and tables produced to maximize factory's profit

☞ Formulation

x_1 : no. of chairs to be produced

x_2 : no. of tables to be produced

$$\begin{aligned} \max\ & 20x_1 + 30x_2 \\ s.t.\ & x_1 + 6x_2 \le 288 \\ & 3x_1 + x_2 \le 99 \\ & x_1 \ge 0, x_2 \ge 0 \end{aligned}$$

- Solution

(ex) Transportation problem

- Warehouses: W_1, W_2, W_3

  Demands: D_1, D_2, D_3, D_4

- Cost Table

From/to	D_1	D_2	D_3	D_4	Supplies
W_1	22	36	24	23	20
W_2	31	19	32	26	30
W_3	25	25	16	22	50
Demands	20	30	30	20	

- Problem

  총 수송비용이 최소화 되는 $W_i \rightarrow D_j$ (i=1,2,3, j=1,2,3,4) 의 수송량은?

- Formulation

  x_{ij} (i = 1, 2, 3, j = 1, 2, 3) : $W_i \rightarrow D_j$ 로의 수송량

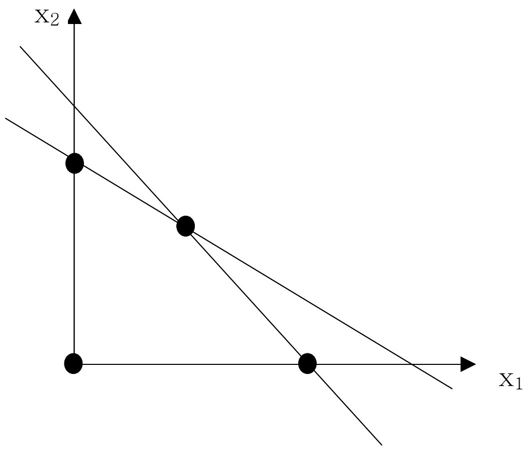
$$\text{Min } u = 22x_{11} + 36x_{12} + 24x_{13} + 23x_{14}$$
$$+31x_{21} + 19x_{22} + 32x_{23} + 26x_{24}$$
$$+25x_{31} + 25x_{32} + 16x_{33} + 22x_{34}$$

s.t

$$x_{11} + x_{21} + x_{31} = 20$$
$$x_{12} + x_{22} + x_{32} = 30$$
$$x_{13} + x_{23} + x_{33} = 30$$
$$x_{14} + x_{24} + x_{34} = 20$$
$$x_{11}, x_{12}, \cdots, x_{33}, x_{34} \geq 0$$

■ Corner point theorem

The maximum or minimum of a linear program, if it exists, will necessarily occur at vertices (corner point) of the constraint set.



■ Simplex Method (單体法)

- LP 의 해를 구하는 컴퓨터 알고리즘

- Local Search

  S1. 임의의 vertex 를 초기해로 시작

  S2. **주변 (neighborhood)** 의 vertex 로 해 이동 => pivoting (선회)

  S3. 더 이상 해가 개선되지 않을 때 까지 S2 반복

- Global optimum solution

  LP ⊂ CP

- Pivoting rule

  주변의 여러 vertices 중 어느 vertex 로 이동할 것인가 ?

- Stop condition

  알고리즘의 종료 조건은 무엇인가 ?

■ Canonical form

- inequality equation => equality equation
- slack variables 추가

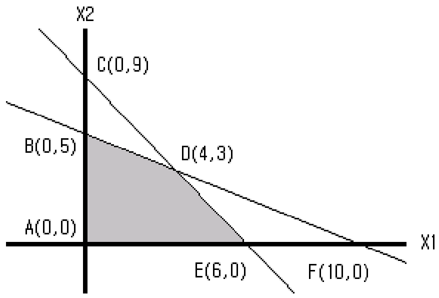
(ex)

Max $z = x_1 + x_2$

s.t $x_1 + 2x_2 \le 10,$

$3x_1 + 2x_2 \le 18,$

$x_1 \ge 0, \quad x_2 \ge 0$



slack variable x_3, x_4 추가 => canonical form

$x_1 + 2x_2 + x_3 = 10$ ---- (1)

$3x_1 + 2x_2 + x_4 = 18$ ---- (2)

$x_1, x_2, x_3, x_4 \ge 0.$

■ Tableau form

$x_3 = -x_1 - 2x_2 + 10$

$x_4 = -3x_1 - 2x_2 + 18$

	x_1	x_2	x_3	x_4	
x_3	-1	-2	0	0	10
x_4	-3	-2	0	0	18
z	1	1	0	0	0

■ Basic Variables / Non-basic variables

- Basic variable

  main constraints equations 의 각 식에 유일하게 존재하는 변수

- Non-basic variable

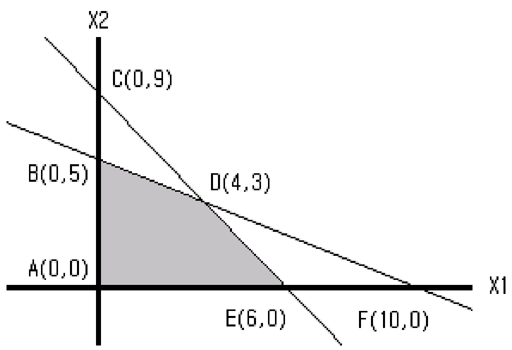
  basic variable 이 아닌 변수

■ Basic Point / Basic Feasible Point

- Basic point : non-basic variables 가 모두 0 인 경우, main constraints eq. 의 해
- Basic feasible point : basic point 중 non-negative constraints 를 만족시키는 해

(ex)

	x_1	x_2	x_3	x_4	(1)	(2)	vertices	z
A	0	0	10	18	x_3=10	x_4=18	O	0
B	0	5	0	8	x_2=5	x_4=8	O	5
C	0	9	-8	0	x_3=-8	x_2=9	X	X
D	4	3	0	0	x_1=4	x_2=3	O	7
E	6	0	4	0	x_3=4	x_1=6	O	6
F	10	0	0	-12	x_1=10	x_4=-12	X	X



■ Corner point theorem (modified)

The maximum of a linear program in canonical form, if it exist, will occur at a basic feasible point of that program.

■ Pivoting

- 현재의 vertex 이웃의 vertex 방문
- 다른 basic point를 방문
- basic variable (non-basic variable) 교체

(ex)

$$\max z = x_1 + x_2 \cdots\cdots (1)$$
$$s.t.\quad x_1 + 2x_2 + x_3 = 10 \cdots\cdots (2)$$
$$3x_1 + 2x_2 + x_4 = 18 \cdots\cdots (3)$$
$$x_1, x_2, x_3, x_4 \geq 0$$

	x_1	x_2	x_3	x_4	
x_3	-1	-2	0	0	10
x_4	-3(*)	-2	0	0	18
z	1	1	0	0	0

current basic variable: x_3, x_4

current basic point: $(x_1, x_2, x_3, x_4) = (0, 0, 10, 18)$ => $z = 0$

pivoting

x_1 : basic variable로 진입

x_4 : basic variable에서 탈락

식(3): $x_1 = -\frac{2}{3}x_2 - \frac{1}{3}x_4 + 6$

->(1), (2)

$$\max z = \frac{1}{3}x_2 - \frac{1}{3}x_4 + 6$$
$$s.t.\quad \frac{4}{3}x_2 + x_3 - \frac{1}{3}x_4 = 4$$
$$x_1 + \frac{2}{3}x_2 + \frac{1}{3}x_4 = 6$$
$$x_1, x_2, x_3, x_4 \geq 0$$

new basic variable: x_3, x_1

new basic point: $(x_1, x_2, x_3, x_4) = (6, 0, 4, 0)$ => $z = 6$

	x_1	x_2	x_3	x_4	
x_3	0	-4/3	0	1/3	4
x_1	0	-2/3	0	-1/3	6
z	0	1/3	0	-1/3	6

■ Pivoting in tableau

	x_i	x_j	x_k	x_l	
x_k	a_{ki}	$a_{kj}^{(*)}$	0	0	b_k
x_l	a_{li}	a_{lj}	0	0	b_l
z	c_i	c_j	0	0	d

↓

	x_i	x_j	x_k	x_l	
x_j	$-a_{ki}/a_{kj}$	0	$1/a_{kj}$	0	$-b_k/a_{kj}$
x_l	$a_{li}-(a_{ki})(a_{lj})/a_{kj}$	0	a_{lj}/a_{kj}	0	$b_l-(b_k)(a_{lj})/a_{kj}$
z	$c_i-(a_{kj})(c_j)/a_{kj}$	0	c_j/a_{kj}	0	$d-(b_k)(c_j)/a_{kj}$

■ Pivoting Element 선정

(1) $c_i > 0$ 인 열 선택

(2) $-b_j/a_{ij}$ 가 최소인 행 선택

■ Stop 조건

모든 c_i 중 양수가 존재하지 않을 때 => optimal solution

(ex)

	x_1	x_2	x_3	x_4	
x_3	-1	-2	0	0	10
x_4	$-3^{(*)}$	-2	0	0	18
z	1	1	0	0	0

↓ pivoting

	x_1	x_2	x_3	x_4	
x_3	0	$-4/3^{(*)}$	0	1/3	4
x_1	0	-2/3	0	-1/3	6
z	0	1/3	0	-1/3	6

↓ pivoting

	x_1	x_2	x_3	x_4	
x_2	0	0	-3/4	1/4	3
x_1	0	0	1/2	-1/2	4
z	0	0	-1/4	-1/4	7

stop

: 최적해 (x_1, x_2) = (3, 4), $z = 7$

(Ex)

$$\max\ 20x_1 + 30x_2$$
$$x_1 + 6x_2 \le 288$$
$$3x_1 + x_2 \le 99$$
$$x_1 \ge 0, x_2 \ge 0$$

canonical form

$$x_1 + 6x_2 + x_3 = 288$$
$$3x_1 + x_2 + x_4 = 99$$
$$x_1 \ge 0, x_2 \ge 0, x_3 \ge 0, x_4 \ge 0$$

$$x_3 = -x_1 - 6x_2 + 288$$
$$x_4 = -3x_1 - x_2 + 99$$
$$x_1 \ge 0, x_2 \ge 0, x_3 \ge 0, x_4 \ge 0$$

tableau

	x_1	x_2	x_3	x_4	
x_3	-1	-6$^{(*)}$	0	0	288
x_4	-3	-1	0	0	99
z	20	30	0	0	0

↓

	x_1	x_2	x_3	x_4	
x_2	-1/6	0	-1/6	0	48
x_4	-17/6$^{(*)}$	0	1/6	0	51
z	15	0	-5	0	1440

↓

	x_1	x_2	x_3	x_4	
x_2	0	0	-3/17	1/17	45
x_1	0	0	-1/17	-6/17	18
z	0	0	-100/17	-90/17	1710