Manipulator Kinematics

Introduction

- **Joint variables**
  - 로봇 각 관절 (joint) 의 위치 값
  - Revolute joint: 회전각도 $(\theta_1, \theta_2, \ldots \ldots)$
    Prismatic joint: 이동변위 $(d_1, d_2, \ldots \ldots)$
  - Joint coordinate 기준

- **Cartesian variables**
  - 로봇 핸드의 위치 값
  - Position: (x, y, z)
    Orientation $(\varphi_x, \varphi_y, \varphi_z)$

$$(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6) \;\; \begin{matrix} \rightarrow \\ \leftarrow \end{matrix} \;\; (x, y, z, \varphi_x, \varphi_y, \varphi_z)$$

Introduction

- **Forward Kinematics**
  - to determine where the robot's hand is?
  - joint variables => Cartesian variables

- **Inverse Kinematics**
  - to calculate what each joint variable is?
  - Cartesian variables => joint variables

$$(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6) \underset{\leftarrow}{\rightarrow} (x, y, z, \varphi_x, \varphi_y, \varphi_z)$$

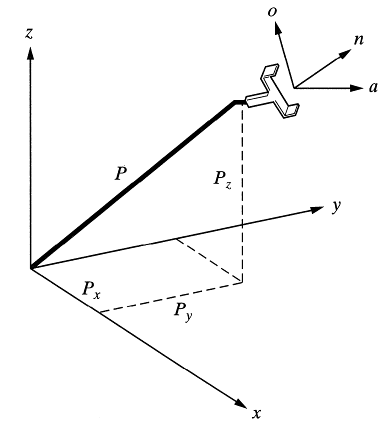
Kinematic Equations: Position

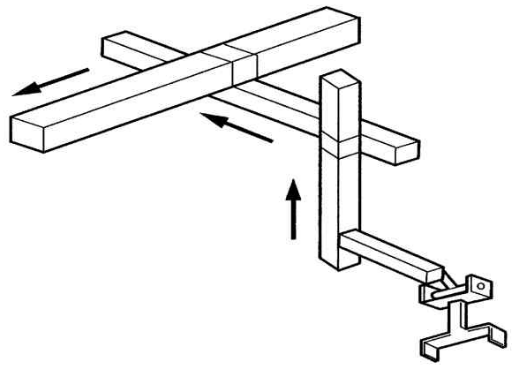
- **Cartesian (gantry, rectangular) coordinates**
  - Forward

$$^{R}T_{P} = Trans(P_x, P_y, P_z)$$

$$^{R}T_{P} = T_{cart} = \begin{bmatrix} 1 & 0 & 0 & P_x \\ 0 & 1 & 0 & P_y \\ 0 & 0 & 1 & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P_x = P_x$$
$$P_y = P_y$$
$$P_z = P_z$$





  - Inverse
    - Joint variables = Cartesian variables

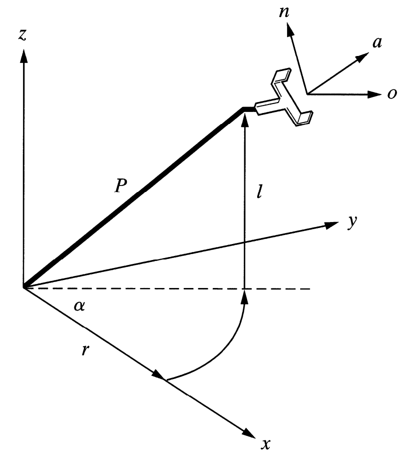
Kinematic Equations : Position

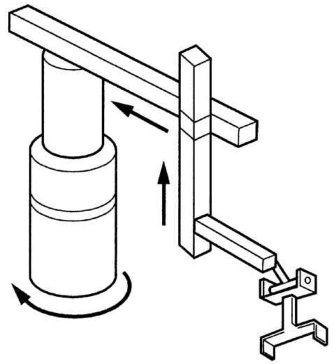
- **Cylindrical coordinates**
  - Forward

$$ {}^{R}T_{P} = T_{cyl}(r,\alpha,l) = \text{Trans}(0,0,l)\text{Rot}(z,\alpha)\text{Trans}(r,0,0) $$

$$ {}^{R}T_{P} = T_{cyl} = \begin{bmatrix} C\alpha & -S\alpha & 0 & rC\alpha \\ S\alpha & C\alpha & 0 & rS\alpha \\ 0 & 0 & 1 & l \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ P_x = r\,C\alpha $$
$$ P_y = r\,S\alpha $$
$$ P_z = l $$





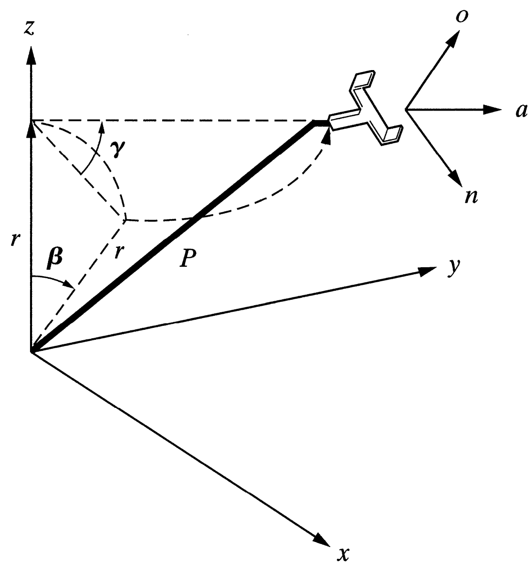
  - Inverse

(ex) $(P_x, P_y, P_z) = (3, 4, 7)$ 일 때 $r, l, \propto =?$

Kinematic Equations : Position

- **Spherical coordinates**
  - Forward

$$^{R}T_{P} = T_{sph}(\gamma, \beta, l) = \mathrm{Rot}(z, \gamma)\mathrm{Rot}(y, \beta)\mathrm{Trans}(0,0,r)$$

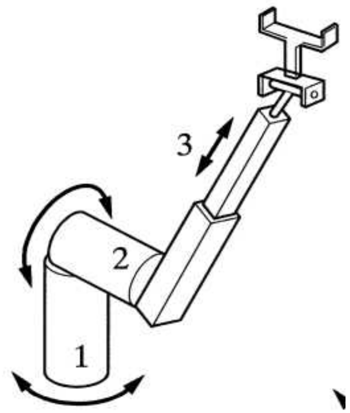


$$^{R}T_{P} = T_{sph} = \begin{bmatrix} C\beta \cdot C\gamma & -S\gamma & S\beta \cdot C\gamma & rS\beta \cdot C\gamma \\ C\beta \cdot S\gamma & C\gamma & S\beta \cdot S\gamma & rS\beta \cdot S\gamma \\ -S\beta & 0 & C\beta & rC\beta \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P_x = r\, S\beta\, C\gamma$$
$$P_y = r\, S\beta\, S\gamma$$
$$P_z = r\, C\beta$$

  - Inverse

(ex) $(P_x, P_y, P_z) = (3, 4, 7)$ 일 때 r, β, γ =?

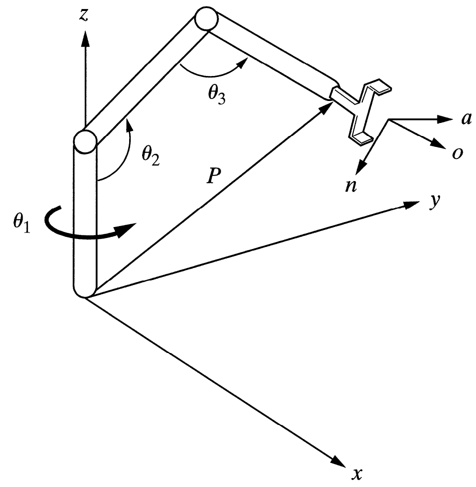


Kinematic Equations : Position

- **Articulated coordinates**
  - Forward

$${}^{R}T_{P} = T_{art}(\theta_1, \theta_2, \theta_3) = \text{Rot}(?, \theta_1)\ \text{Rot}(?, \theta_2)\ \text{Rot}(?, \theta_3)$$

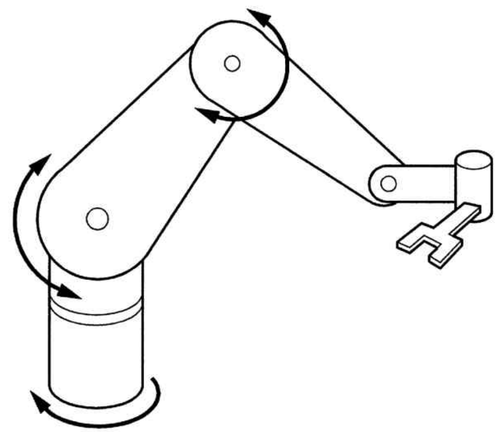
$$P_x = f_1(\theta_1, \theta_2, \theta_3) = ?$$
$$P_y = f_2(\theta_1, \theta_2, \theta_3) = ??$$
$$P_z = f_3(\theta_1, \theta_2, \theta_3) = ???$$



  - ➢ Denavit-Hartenberg representation

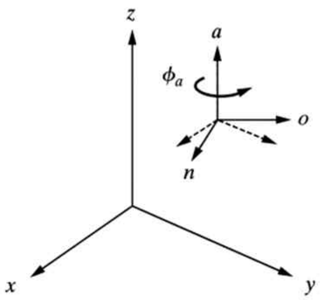
  - Inverse

(ex) $(P_x, P_y, P_z) = (3, 4, 7)$ 일 때 $\theta_1, \theta_2, \theta_3 = ?$

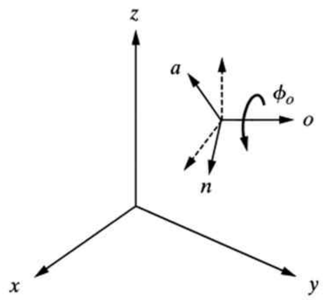


Kinematic Equations : Orientation

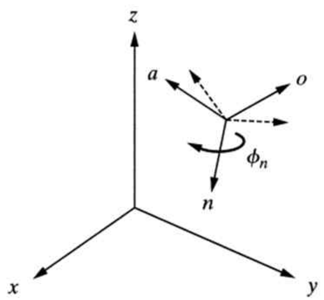
- **Roll, Pitch, Yaw (RPY) angles**
  1) Roll: Rotation of ϕ_a about $\bar{a}$-axis (z-axis of the moving frame)
  2) Pitch: Rotation of ϕ_o about $\bar{o}$-axis (y-axis of the moving frame)
  3) Yaw: Rotation of ϕ_n about $\bar{n}$-axis (x-axis of the moving frame)



Rotation about ϕ_a



Rotation about ϕ_o



Rotation about ϕ_n

$$RPY(\phi_a,\phi_o,\phi_n) = Rot(a,\phi_a)\cdot Rot(o,\phi_o)\cdot Rot(n,\phi_n)$$

$$= \begin{bmatrix} C\phi_a C\phi_o & C\phi_a S\phi_o S\phi_n - S\phi_a C\phi_n & C\phi_a S\phi_o C\phi_n + S\phi_a S\phi_n & 0 \\ S\phi_a C\phi_o & S\phi_a S\phi_o S\phi_n + C\phi_a C\phi_n & S\phi_a S\phi_o C\phi_n - C\phi_a S\phi_n & 0 \\ -S\phi_a & C\phi_a S\phi_n & C\phi_o C\phi_n & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Kinematic Equations : Orientation

- Transformation Matrix 에서 RPY angle 구하는 방법

1) $RPY(\phi_a,\phi_o,\phi_n) = Rot(a,\phi_a)\cdot Rot(o,\phi_o)\cdot Rot(n,\phi_n)$

$$= \begin{bmatrix} n_x & o_x & a_x & 0 \\ n_y & o_y & a_y & 0 \\ n_z & o_z & a_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2) $$Rot(a,\phi_a)^{-1} \begin{bmatrix} n_x & o_x & a_x & 0 \\ n_y & o_y & a_y & 0 \\ n_z & o_z & a_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = Rot(o,\phi_o)\cdot Rot(n,\phi_n)$$

3) $$\begin{bmatrix} n_xC\phi_a + n_yS\phi_a & o_xC\phi_a + o_yS\phi_a & a_xC\phi_a + a_yS\phi_a & 0 \\ n_yC\phi_a - n_xS\phi_a & o_yC\phi_a - o_xS\phi_a & a_yC\phi_a - a_xS\phi_a & 0 \\ n_z & o_z & a_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} C\phi_o & S\phi_oS\phi_n & S\phi_oC\phi_n & 0 \\ 0 & C\phi_n & -S\phi_n & 0 \\ -S\phi_o & C\phi_oS\phi_n & C\phi_oC\phi_n & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4) $n_yC\phi_a - n_xS\phi_a = 0$

$\Rightarrow$ $\phi_a = ATAN2(n_y, n_x)$, $\phi_a = ATAN2(-n_y, -n_x)$

$S\phi_o = -n_z$, $C\phi_o = n_xC\phi_a + n_yS\phi_a$

$\Rightarrow$ $\phi_o = ATAN2(-n_z, n_xC\phi_a + n_yS\phi_a)$

$C\phi_n = o_yC\phi_a + o_xS\phi_a$, $S\phi_n = -a_yC\phi_a + a_xS\phi_a$

$\Rightarrow$ $\phi_n = ATAN2(-a_yC\phi_a + a_xS\phi_a, o_yC\phi_a - o_xS\phi_a)$

★ $\theta = ATAN2(y, x)$

$\theta = \mathrm{atan}\left(\frac{y}{x}\right)$, $-90 \le \theta \le 90$

$\theta = \mathrm{atan2}(y, x)$, $-180 \le \theta \le 180$

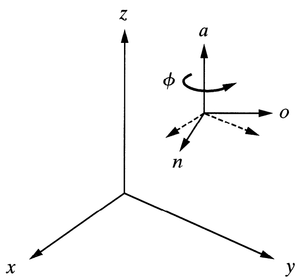
Kinematic Equations : Orientation

(ex) $\mathrm{T} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ → roll, pitch, yaw ?

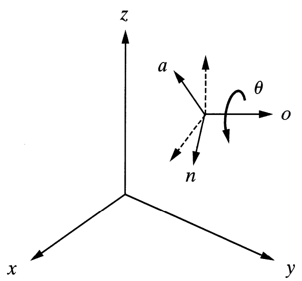
(ex) $\mathrm{T} = \begin{bmatrix} 0.354 & -0.674 & 0.649 & 4.33 \\ 0.505 & 0.722 & 0.475 & 2.50 \\ -0.788 & 0.160 & 0.595 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ → $P_x, P_y, P_z, \phi_a, \phi_o, \phi_n$?

Kinematic Equations : Orientation

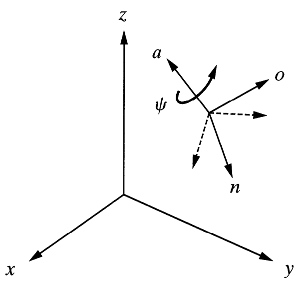
- **Euler angles**
  1) Rotation of ϕ about $\bar{a}$-axis (z-axis of the moving frame)
  2) Rotation of θ about $\bar{o}$-axis (y-axis of the moving frame)
  3) Rotation of ψ about $\bar{a}$-axis (z-axis of the moving frame)



Rotation of ϕ about the a-axis



Rotation of θ about the o-axis



Rotation of ψ about the a-axis

$$Euler(\phi,\theta,\psi) = Rot(a,\phi)\cdot Rot(o,\theta)\cdot Rot(a,\psi)$$

$$= \begin{bmatrix} C\phi C\theta C\psi - S\phi S\psi & -C\phi C\theta S\psi - S\phi C\psi & C\phi S\theta & 0 \\ S\phi C\theta C\psi + C\phi S\psi & -S\phi C\theta S\psi + C\phi C\psi & S\phi S\theta & 0 \\ -S\theta C\psi & S\theta S\psi & C\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Kinematic Equations : Orientation

- Transformation Matrix 에서 Euler angle 구하는 방법

$$Euler(\phi,\theta,\psi) = \begin{bmatrix} n_x & o_x & a_x & 0 \\ n_y & o_y & a_y & 0 \\ n_z & o_z & a_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\phi = ATAN2(a_y, a_x),\ \phi = ATAN2(-a_y, -a_x)$$

$$\psi = ATAN2(-n_x S\phi + n_y C\phi, -o_x S\phi + o_y C\phi)$$

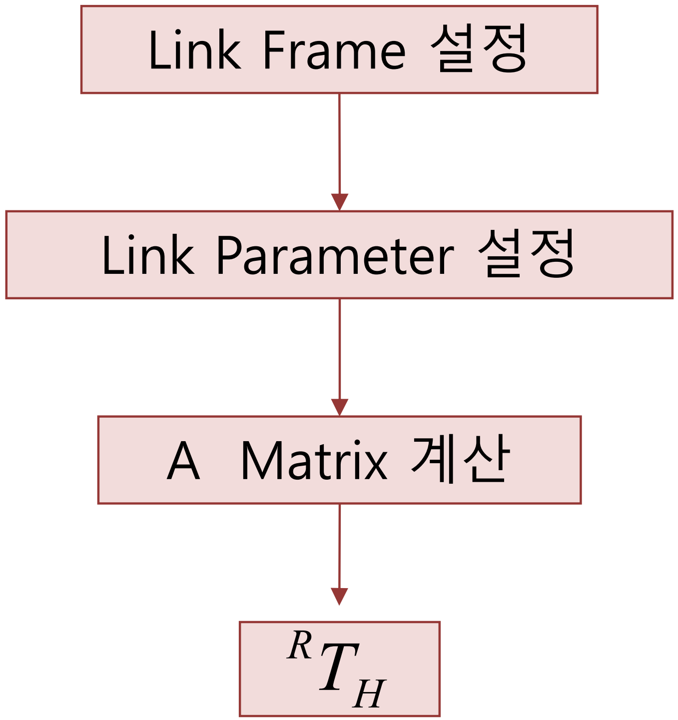
$$\theta = ATAN2(a_x C\phi + a_y S\phi, a_z)$$

Kinematic Equations : Orientation

(ex) $\mathrm{T} = \begin{bmatrix} 0.579 & -0.548 & -0.604 & 5 \\ 0.540 & 0.813 & -0.220 & 7 \\ 0.611 & -0.199 & 0.766 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \ P_x, P_y, P_z, \phi, \theta, \psi?$

DH Representation

- Denavit-Hartenberg
  - robot 의 forward kinematics 를 유도하기 위한 modeling 방법 (1955)
  - 임의 형상 (configuration) 을 갖는 로봇에 대하여 $^{R}T_{H}$ 를 구하는 체계적 방법



DH Representation

1. Link Frame 설정

- Joint 번호, Link 번호
  - base 에서 멀어지는 순으로 1, 2, 3 , …

- 각 link 에 하나의 frame 설정
  - Distal joint 에 설정
  - {0} frame (base frame) 은 world frame 과 평행하게 joint 1 에 설정

S1. z 축 설정
- revolute joint: 회전중심방향
- prismatic joint: 직선운동방향 (멀어지는 방향)

S2. Common Normal (공통수직선) 설정
- 인접한 두 joint 축 (Zn-1, Zn) 에 모두 수직인 직선
- Case I: 1개 존재, Case II: ∞개 존재, Case III: 0개 존재

S3. X 축 설정
- Case I: common normal 방향
- Case II: common normal 방향 (link 중심을 지나는 직선)
- Case III: Zn-1 축과 Zn 축이 이루는 평면에 직교 방향

S4. 원점 설정
- Case I, II: Xn 축과 Zn 축의 교점
- Case III: Zn-1 축과 Zn 축의 교점

DH Representation

2. Link Parameter 설정

a_n: link length

distance, along the common normal (X_n), between the joint axes (Z_{n-1}, Z_n)

α_n: twist angle

angle, along the common normal (X_n), between the joint axes (Z_{n-1}, Z_n)

d_n: distance between links

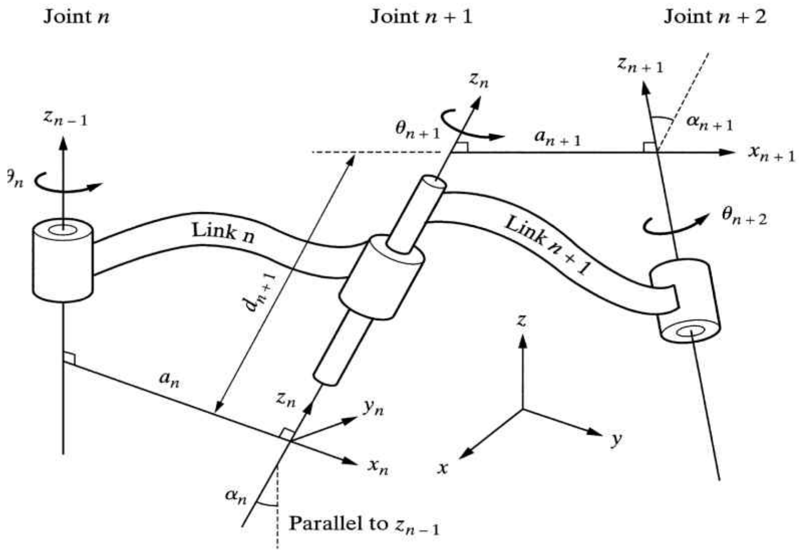
distance, along the joint axis (Z_{n-1}), between the intersections of common normals with joint axes ($X_{n-1} \perp Z_{n-1}$) ∧ ($X_n \perp Z_n$),

* prismatic joint 의 variable

θ_n: angle between links

angle, along the joint axis (Z_{n-1}), between the common normals (X_{n-1}, X_n)

* revolute joint 의 variable



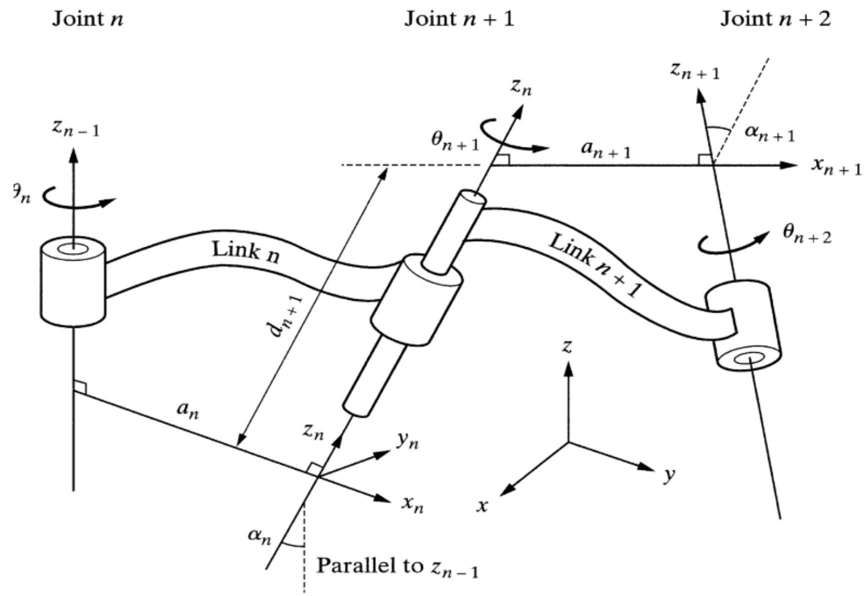
DH Representation

3. A Matrix

A_n = ${}^{n-1}T_n$: {n-1} frame 과 {n} frame 사이의 변환행렬

$$A_n = Rot(\overline{a}, \theta_n) \cdot Trans(0,0,d_n) \cdot Trans(a_n,0,0) \cdot Rot(\overline{n}, \alpha_n)$$

$$= \begin{bmatrix} C\theta_n & -S\theta_n C\alpha_n & S\theta_n S\alpha_n & a_n C\theta_n \\ S\theta_n & C\theta_n C\alpha_n & -C\theta_n S\alpha_n & a_n S\theta_n \\ 0 & S\alpha_n & C\alpha_n & d_n \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



DH Representation

4. $^{R}T_{H}$ matrix

$$
\begin{aligned}
{}^{R}T_{H} &= {}^{0}T_{1} \cdot {}^{1}T_{2} \cdots {}^{n-1}T_{n} \\
&= A_{1} \cdot A_{2} \cdots A_{n} \\
&= \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

DH Representation

5. Forward Kinematics

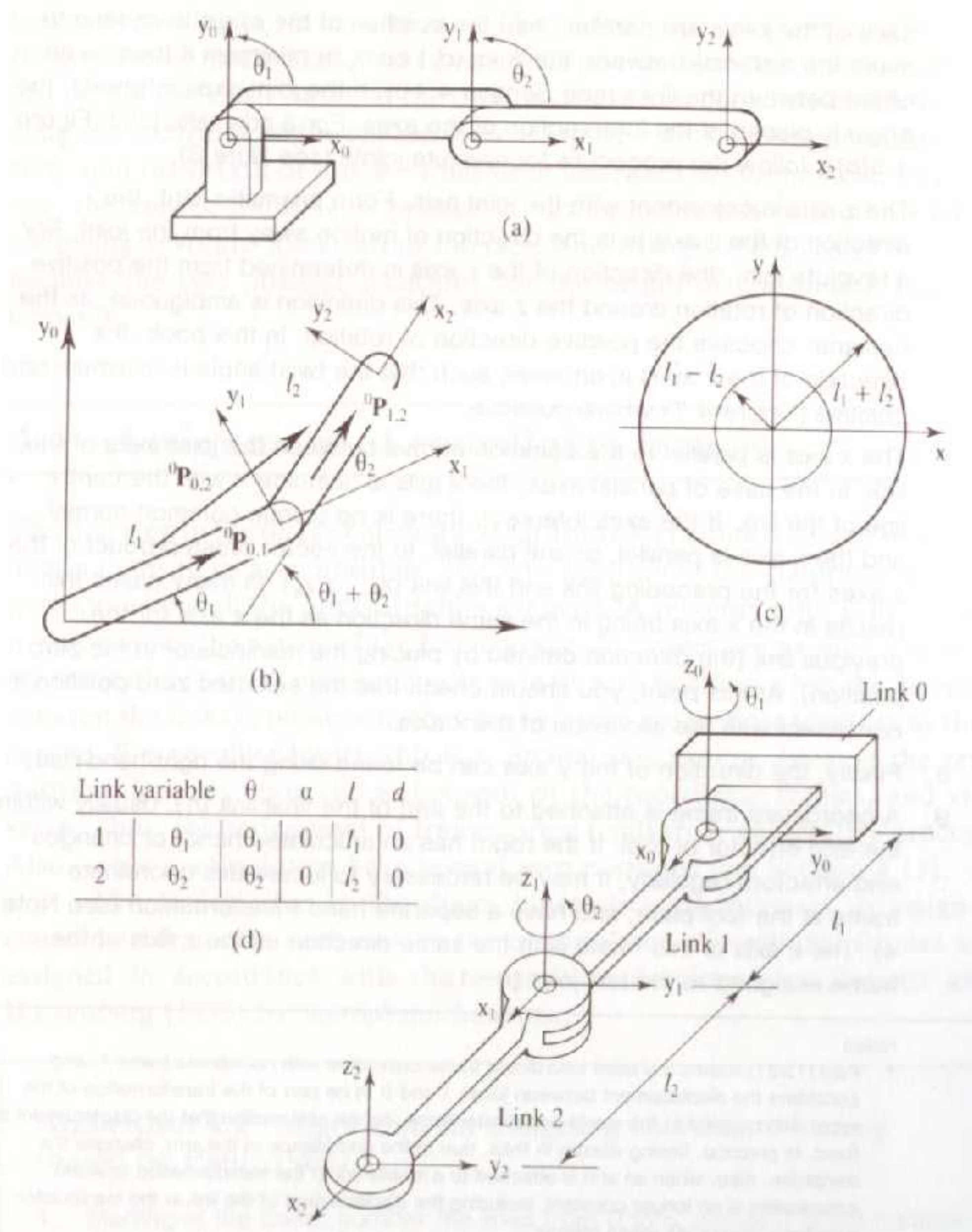
$$x = p_x\,, \quad y = p_y\,, \quad z = p_z$$

$$\phi_a = ATAN2(n_y, n_x)$$

$$\phi_o = ATAN2(-n_z, n_x C\phi_a + n_y S\phi_a)$$

$$\phi_n = ATAN2(-a_y C\phi_a + a_x S\phi_a, o_y C\phi_a + o_x S\phi_a)$$

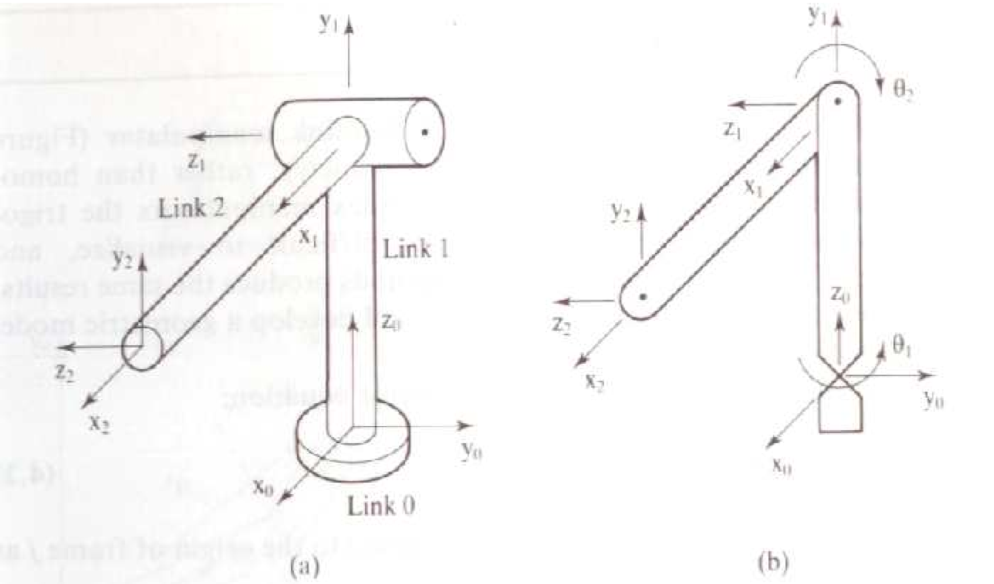
Type 1

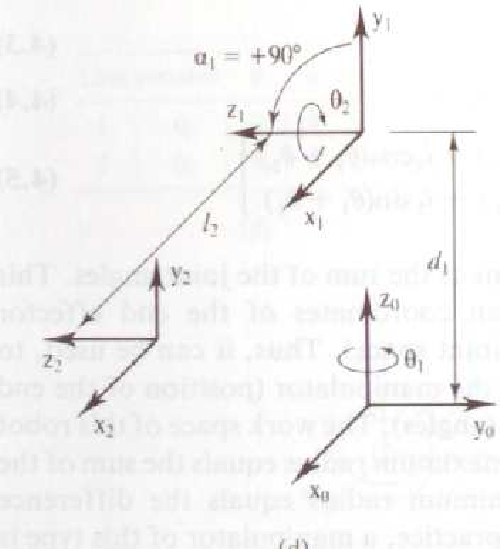


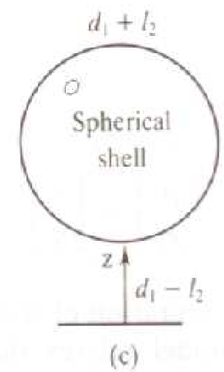
Link variable	θ	α	l	d
1	θ_1	0	l_1	0
2	θ_2	0	l_2	0

(d)

Type 2

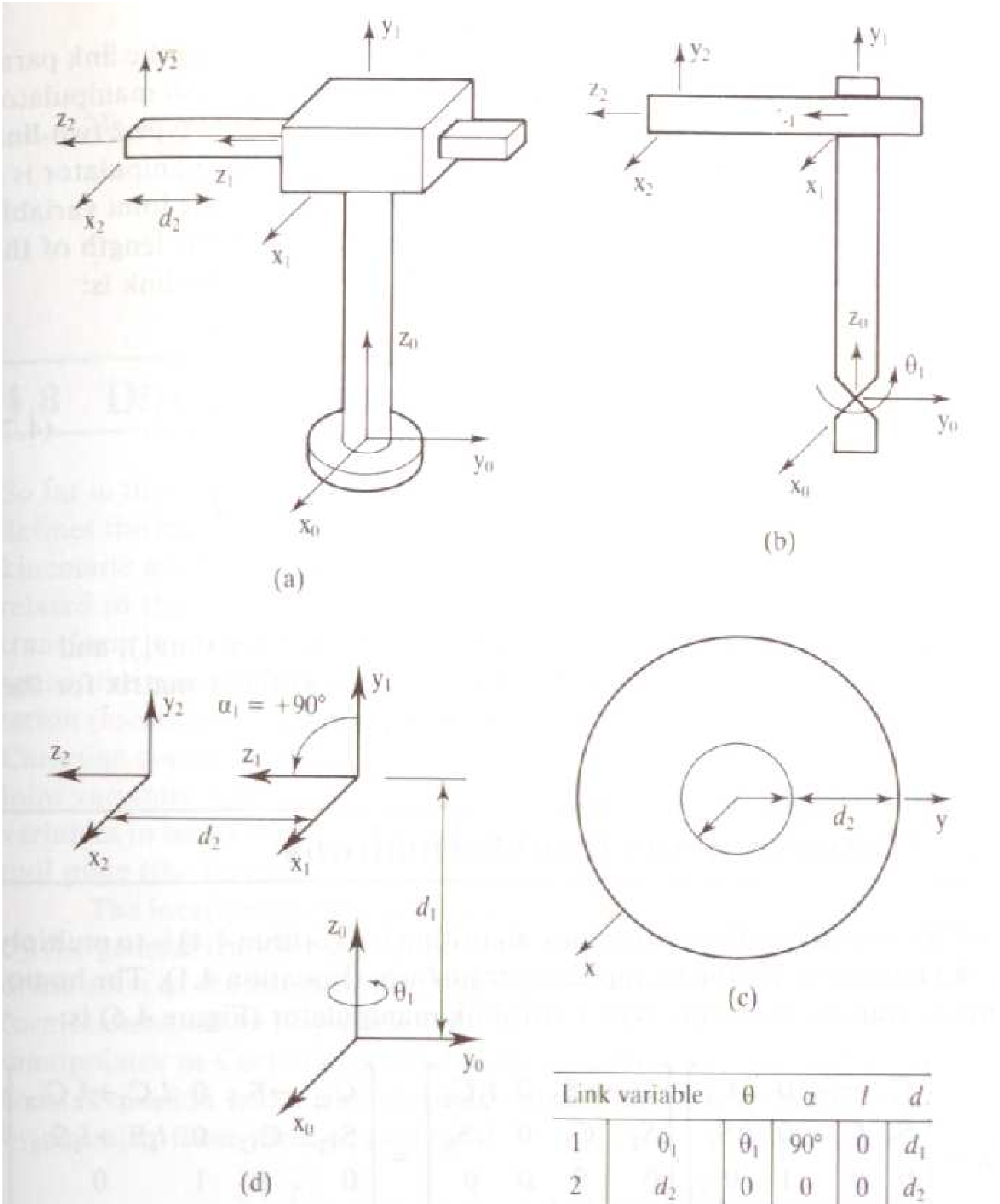






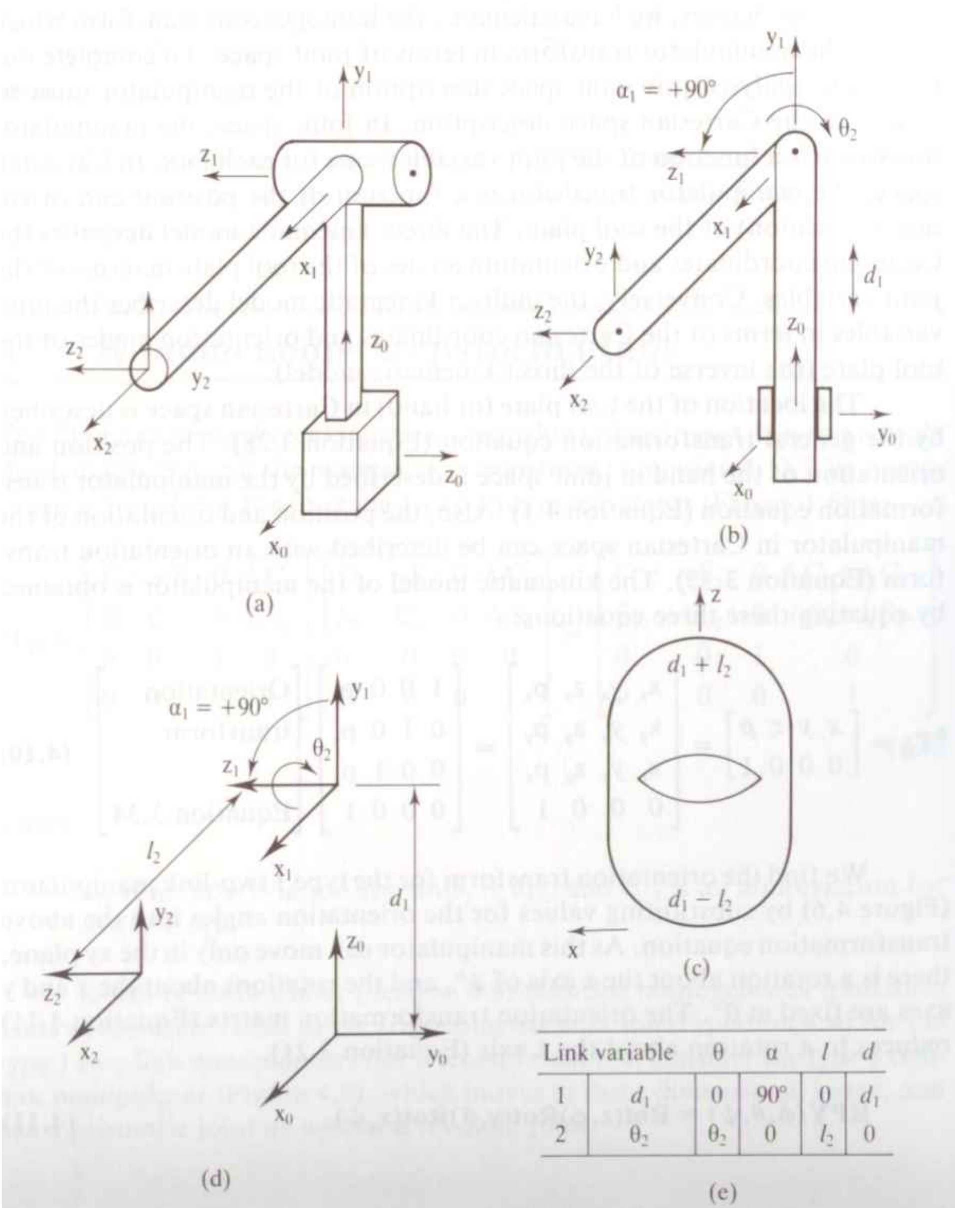
Link	variable	θ	α	l	d
1	θ_1	θ_1	90°	0	d_1
2	θ_2	θ_2	0	l_2	0

Type 3



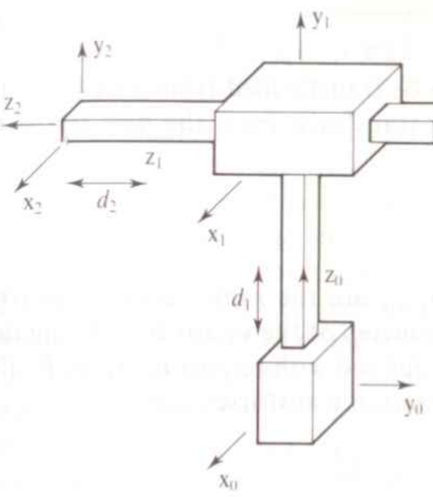
Link	variable	θ	α	l	d
1	θ_1	θ_1	90°	0	d_1
2	d_2	0	0	0	d_2

Type 4

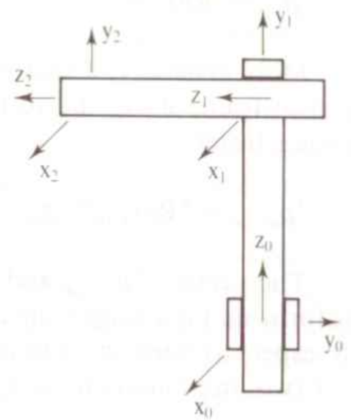


Link variable	θ	α	l	d
1	d_1	0	90°	0	d_1
2	θ_2	θ_2	0	l_2	0

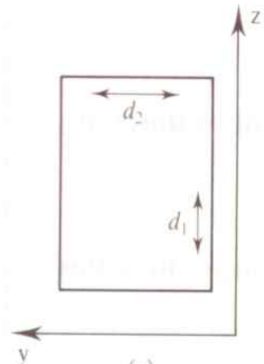
Type 5



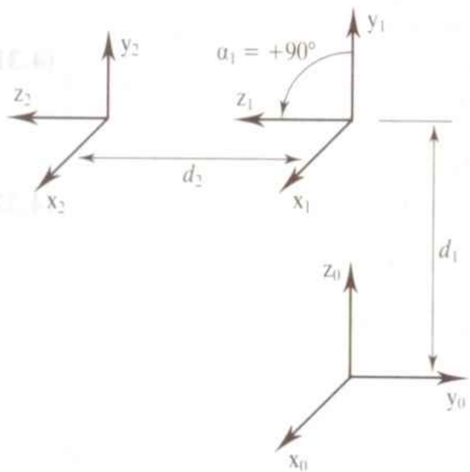
(a)



(b)



(c)

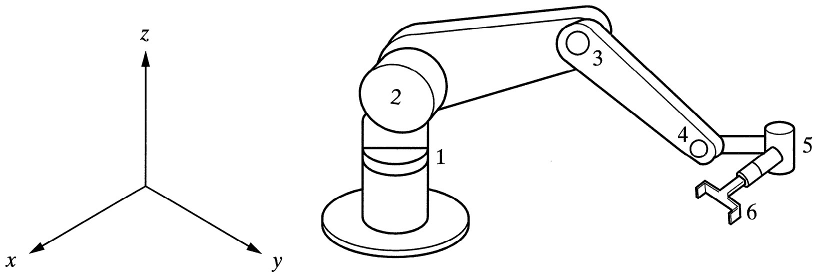


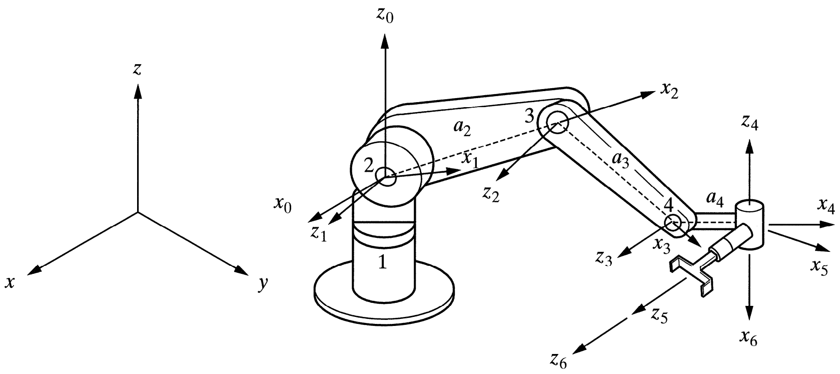
(d)

Link	variable	θ	α	l	d
1	d_1	0	90°		d_1
2	d_2	0	0	0	d_2

(e)

Articulate Robot





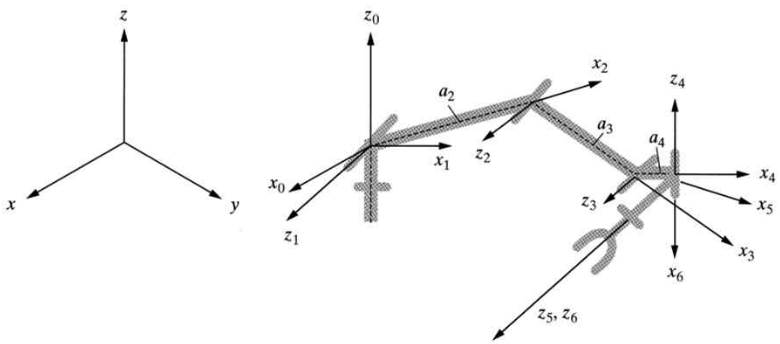


TABLE 2.2 PARAMETERS FOR THE ROBOT OF EXAMPLE 2.19

#	θ	d	a	α
1	θ_1	0	0	90
2	θ_2	0	a_2	0
3	θ_3	0	a_3	0
4	θ_4	0	a_4	−90
5	θ_5	0	0	90
6	θ_6	0	0	0

Articulate Robot

$$A_1 = \begin{bmatrix} C_1 & 0 & S_1 & 0 \\ S_1 & 0 & -C_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} C_2 & -S_2 & 0 & C_2 a_2 \\ S_2 & C_2 & 0 & S_2 a_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} C_3 & -S_3 & 0 & C_3 a_3 \\ S_3 & C_3 & 0 & S_3 a_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad A_4 = \begin{bmatrix} C_4 & 0 & -S_4 & C_4 a_4 \\ S_4 & 0 & C_4 & S_4 a_4 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad (2.55)$$

$$A_5 = \begin{bmatrix} C_5 & 0 & S_5 & 0 \\ S_5 & 0 & -C_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad A_6 = \begin{bmatrix} C_6 & -S_6 & 0 & 0 \\ S_6 & C_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$^{R}T_H = A_1 A_2 A_3 A_4 A_5 A_6 \qquad (2.57)$$

$$= \begin{bmatrix} C_1(C_{234}C_5C_6 - S_{234}S_6) - S_1S_5C_6 & C_1(-C_{234}C_5C_6 - S_{234}C_6) + S_1S_5S_6 & C_1(C_{234}S_5) + S_1C_5 & C_1(C_{234}a_4 + C_{23}a_3 + C_2a_2) \\ S_1(C_{234}C_5C_6 - S_{234}S_6) + C_1S_5S_6 & S_1(-C_{234}C_5C_6 - S_{234}C_6) - C_1S_5S_6 & S_1(C_{234}S_5) - C_1C_5 & S_1(C_{234}a_4 + C_{23}a_3 + C_2a_2) \\ S_{234}C_5C_6 + C_{234}S_6 & -S_{234}C_5C_6 + C_{234}C_6 & S_{234}S_5 & S_{234}a_4 + S_{23}a_3 + S_2a_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Stanford Arm

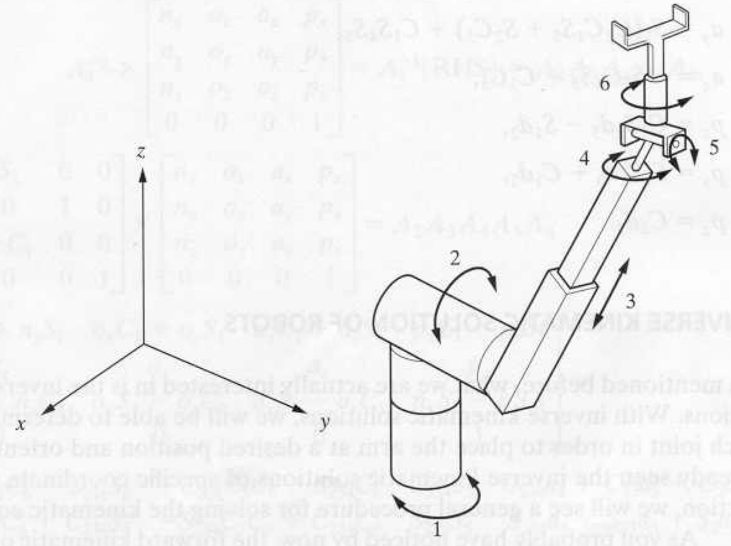


Figure 2.29 Schematic drawing of the Stanford Arm.

$$
{}^{R}T_{H_{\text{STANFORD}}} = {}^{0}T_{6} = \begin{bmatrix} n_x & o_x & a_x & P_x \\ n_y & o_y & a_y & P_y \\ n_z & o_z & a_z & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix},
$$

$$
\begin{aligned}
n_x &= C_1[C_2(C_4C_5C_6 - S_4S_6) - S_2S_5C_6] - S_1(S_4C_5C_6 + C_4S_6),\\
n_y &= S_1[C_2(C_4C_5C_6 - S_4S_6) - S_2S_5C_6] + C_1(S_4C_5C_6 + C_4S_6),\\
n_z &= -S_2(C_4C_5C_6 - S_4S_6) - C_2S_5C_6,\\
o_x &= C_1[-C_2(C_4C_5C_6 + S_4C_6) + S_2S_5S_6] - S_1(-S_4C_5S_6 + C_4C_6),\\
o_y &= S_1[-C_2(C_4C_5S_6 + S_4C_6) + S_2S_5S_6] + C_1(-S_4C_5S_6 + C_4C_6),\\
o_z &= S_2(C_4C_5S_6 + S_4C_6) + C_2S_5S_6,\\
a_x &= C_1(C_2C_4S_5 + S_2C_5) - S_1S_4S_5,\\
a_y &= S_1(C_2C_4S_5 + S_2C_5) + C_1S_4S_5,\\
a_z &= -S_2C_4S_5 + C_2C_5,\\
p_x &= C_1S_2d_3 - S_1d_2,\\
p_y &= S_1S_2d_3 + C_1d_2,\\
p_z &= C_2d_3.
\end{aligned}
$$

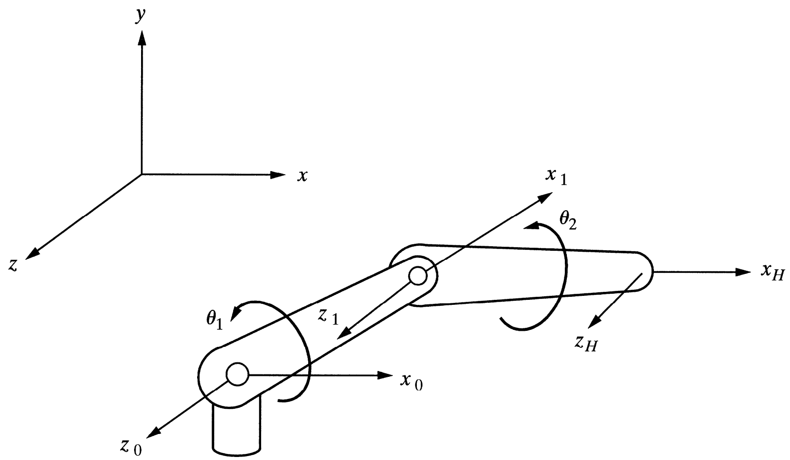
Inverse Kinematic Solutions

Joint Variables $(\theta_1, \theta_2, \cdots, \theta_n)$ $\xleftarrow{\text{I.K}}$ Cartesian Variables $(p_x, p_y, p_z, \phi_a, \phi_o, \phi_n)$

- 여러 개의 해가 존재 : redundancy
  * degeneracy : 무한 개의 해가 존재하는 경우
- 일반적 해법이 없다: heuristic solution
  - Geometric approach
  - Analytic approach

Inverse Kinematic Solutions

(Example 1)

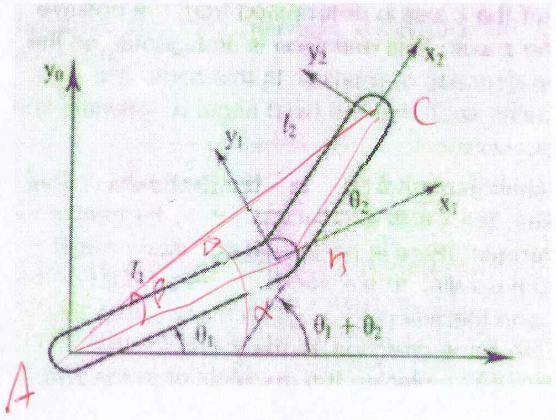


Given: p_x, p_y

Find: θ_1, θ_2

Inverse Kinematic Solutions

1. Geometric approach



$$p_x^{\ 2} + p_y^{\ 2} = l_1^{\ 2} + l_2^{\ 2} - 2 l_1 l_2 \cos(180 - \theta_2)$$

$$= l_1^{\ 2} + l_2^{\ 2} + 2 l_1 l_2 C_2$$

$$\therefore C_2 = \frac{p_x^{\ 2} + p_y^{\ 2} - l_1^{\ 2} - l_2^{\ 2}}{2 l_1 l_2}$$

$$S_2 = \pm\sqrt{1 - C_2^{\ 2}}$$

$$\theta_2 = \operatorname{atan2}(S_2, C_2)$$

$$\theta_1 = \alpha - \beta$$

$$\alpha = \operatorname{atan2}(p_y, p_x)$$

$$\beta = \operatorname{atan2}(l_2 S_2\ ,\ l_1 + l_2 C_2)$$

Inverse Kinematic Solutions

2. Algebraic solution (analytic approach)

$$
{}^{0}T_{H} = A_1 \times A_2 = \begin{bmatrix} C_{12} & -S_{12} & 0 & a_2 C_{12} + a_1 C_1 \\ S_{12} & C_{12} & 0 & a_2 S_{12} + a_1 S_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.56}
$$

$$S_{12} = n_y \text{ and } C_{12} = n_x \rightarrow \theta_{12} = ATAN2(n_y, n_x)$$

$$a_2 C_{12} + a_1 C_1 = p_x \text{ or } a_2 n_x + a_1 C_1 = p_x \rightarrow C_1 = \frac{p_x - a_2 n_x}{a_1}$$

$$a_2 S_{12} + a_1 S_1 = p_y \text{ or } a_2 n_y + a_1 S_1 = p_y \rightarrow S_1 = \frac{p_y - a_2 n_y}{a_1}$$

$$\theta_1 = ATAN2(S_1, C_1) = ATAN2\left(\frac{p_y - a_2 n_y}{a_1}, \frac{p_x - a_2 n_x}{a_1}\right)$$

Inverse Kinematic Solutions

3. Alternative solution (analytic approach)

$$
{}^{0}T_H = A_1 \times A_2 = \begin{bmatrix} C_{12} & -S_{12} & 0 & a_2C_{12} + a_1C_1 \\ S_{12} & C_{12} & 0 & a_2S_{12} + a_1S_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
A_1 \times A_2 \times A_2^{-1} = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \times A_2^{-1} \quad \text{or} \quad A_1 = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \times A_2^{-1}
$$

$$
\begin{bmatrix} C_1 & -S_1 & 0 & a_1C_1 \\ S_1 & C_1 & 0 & a_1S_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} C_2 & -S_2 & 0 & -a_2 \\ S_2 & C_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

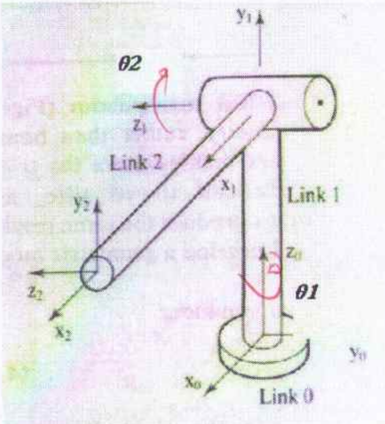
$$
\begin{bmatrix} C_1 & -S_1 & 0 & a_1C_1 \\ S_1 & C_1 & 0 & a_1S_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C_2n_x - S_2o_x & S_2n_x + C_2o_x & a_x & p_x - a_2n_x \\ C_2n_y - S_2o_y & S_2n_y + C_2o_y & a_y & p_y - a_2n_y \\ C_2n_z - S_2o_z & S_2n_z + C_2o_z & a_z & p_z - a_2n_z \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
\theta_1 = ATAN2(S_1, C_1) = ATAN2\left(\frac{p_y - a_2n_y}{a_1}, \frac{p_x - a_2n_x}{a_1}\right)
$$

Inverse Kinematic Solutions

(Example 2)

◆ Analytic Approach



Step1. Forward Kinematics

$$
{}^{R}T_H = \begin{bmatrix} C_1C_2 & -C_2S_2 & S_1 & l_2C_1C_2 \\ S_1S_2 & -S_1S_2 & -C_1 & l_2S_1C_2 \\ S_2 & C_2 & 0 & l_1+l_2S_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
= \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Step 2:

$$
\frac{p_y}{p_x} = \frac{S_1}{C_1}
$$

$$
\therefore \ \theta_1 = \mathrm{atan2}(p_y, p_x)\,,\ \mathrm{atan2}(-p_y, -p_x)
$$

Step 3: ${}^{R}T_H = A_1 \cdot A_2$

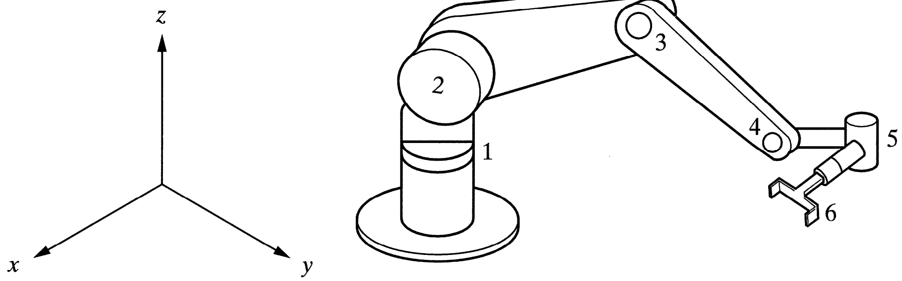
$$
\therefore A_1^{-1} \cdot {}^{R}T_H = A_2
$$

$$
\begin{bmatrix} n_xC_1+n_yS_1 & o_xC_1+o_yS_1 & a_xC_1+a_yS_1 & p_xC_1+p_yS_1 \\ n_z & o_z & a_z & p_z-l_1 \\ n_xS_1-n_yC_1 & o_xS_1-o_yC_1 & a_xS_1-a_yC_1 & p_xS_1-p_yC_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}
= \begin{bmatrix} C_2 & -S_2 & 0 & l_2C_2 \\ S_2 & C_2 & 0 & l_2C_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
\theta_2 = \mathrm{atan2}(p_z - l_1\,,\ p_xC_1 + p_yS_1)
$$

Inverse Kinematic Solutions

(Example 3)



Forward Kinematics

$$^{R}T_H = A_1A_2A_3A_4A_5A_6$$

$$= \begin{bmatrix} C_1(C_{234}C_5C_6 - S_{234}S_6) - S_1S_5C_6 & C_1(-C_{234}C_5C_6 - S_{234}C_6) + S_1S_5S_6 & C_1(C_{234}S_5) + S_1C_5 & C_1(C_{234}a_4 + C_{23}a_3 + C_2a_2) \\ S_1(C_{234}C_5C_6 - S_{234}S_6) + C_1S_5C_6 & S_1(-C_{234}C_5C_6 - S_{234}C_6) - C_1S_5S_6 & S_1(C_{234}S_5) - C_1C_5 & S_1(C_{234}a_4 + C_{23}a_3 + C_2a_2) \\ S_{234}C_5C_6 + C_{234}S_6 & -S_{234}C_5C_6 + C_{234}C_6 & S_{234}S_5 & S_{234}a_4 + S_{23}a_3 + S_2a_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$^{R}T_H = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Inverse Kinematics

$$A_1^{-1}\times\begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = A_1^{-1}[\text{RHS}] = A_2A_3A_4A_5A_6,$$

$$\begin{bmatrix} C_1 & S_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ S_1 & -C_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\times\begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = A_2A_3A_4A_5A_6$$

$$\begin{bmatrix} n_xC_1+n_yS_1 & o_xC_1+o_yS_1 & a_xC_1+a_yS_1 & p_xC_1+p_yS_1 \\ n_z & o_z & a_z & p_z \\ n_xS_1-n_yC_1 & o_xS_1-o_yC_1 & a_xS_1-a_yC_1 & p_xS_1-p_yC_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} C_{234}C_5C_6-S_{234}S_6 & -C_{234}C_5C_6-S_{234}C_6 & C_{234}S_5 & C_{234}a_4+C_{23}a_3+C_2a_2 \\ S_{234}C_5C_6+C_{234}S_6 & -S_{234}C_5C_6+C_{234}C_6 & S_{234}S_5 & S_{234}a_4+S_{23}a_3+S_2a_2 \\ -S_5C_6 & S_5S_6 & C_5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

From the 3,4 elements of the equation,

$$p_xS_1-p_yC_1=0 \quad\rightarrow\quad \boxed{\theta_1=\tan^{-1}\left(\frac{p_y}{p_x}\right) \quad\text{and}\quad \theta_1=\theta_1+180^\circ.} \qquad (2.62)$$

From the 1,4 and 2,4 elements, we get

$$\begin{aligned} p_xC_1+p_yS_1 &= C_{234}a_4+C_{23}a_3+C_2a_2, \\ p_z &= S_{234}a_4+S_{23}a_3+S_2a_2. \end{aligned} \qquad (2.63)$$

Rearranging the two equations, squaring them, and then adding the squares gives

$$\begin{aligned} &(p_xC_1+p_yS_1-C_{234}a_4)^2=(C_{23}a_3+C_2a_2)^2 \\ &(p_z-S_{234}a_4)^2=(S_{23}a_3+S_2a_2)^2 \\ \hline &(p_xC_1+p_yS_1-C_{234}a_4)^2+(p_z-S_{234}a_4)^2=a_2^2+a_3^2+2a_2a_3(S_2S_{23}+C_2C_{23}). \end{aligned}$$

Referring to the trigonometric functions of Equation (2.56), we see that

$$S_2S_{23}+C_2C_{23}=\cos[(\theta_2+\theta_3)-\theta_2]=\cos\theta_3.$$

Thus,

$$C_3=\frac{(p_xC_1+p_yS_1-C_{234}a_4)^2+(p_z-S_{234}a_4)^2-a_2^2-a_3^2}{2a_2a_3}. \qquad (2.64)$$

In this equation, everything is known except for S_{234} and C_{234}, which we will find next. Knowing that $S_3=\pm\sqrt{1-C_3^2}$, we can then say that

$$\boxed{\theta_3=\tan^{-1}\frac{S_3}{C_3}.} \qquad (2.65)$$

Since joints 2,3, and 4 are parallel, premultiplying by inverses of A_2 and A_3 will not yield useful results. The next step is to premultiply by the inverses of A_1 through A_4, which results in:

$$A_4^{-1}A_3^{-1}A_2^{-1}A_1^{-1}\times\begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = A_4^{-1}A_3^{-1}A_2^{-1}A_1^{-1}[\text{RHS}]=A_5A_6, \qquad (2.66)$$

which, in turn, yields

$$\begin{bmatrix} C_{234}(C_1n_x+S_1n_y)+S_{234}n_z & C_{234}(C_1o_x+S_1o_y)+S_{234}o_z & C_{234}(C_1a_x+S_1a_y)+S_{234}a_x & C_{234}(C_1p_x+S_1p_y)+S_{234}p_z-C_{34}a_2-C_4a_3-a_4 \\ C_1n_y-S_1n_x & C_1o_y-S_1o_x & C_1a_y-S_1a_x & 0 \\ -S_{234}(C_1n_x+S_1n_y)+C_{234}n_z & -S_{234}(C_1o_x+S_1o_y)+C_{234}o_z & -S_{234}(C_1a_x+S_1a_y)+C_{234}a_z & -S_{234}(C_1p_x+S_1p_y)+C_{234}p_z+S_{34}a_2+S_4a_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} C_5C_6 & -C_5S_6 & S_5 & 0 \\ S_5C_6 & -S_5S_6 & -C_5 & 0 \\ S_6 & C_6 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \qquad (2.67)$$

Inverse Kinematics (continued)

From the 3,3 elements of the matrices in Equation (2.67),

$$-S_{234}(C_1a_x + S_1a_y) + C_{234}a_z = 0 \quad \rightarrow$$

$$\theta_{234} = \tan^{-1}\left(\frac{a_z}{C_1a_x + S_1a_y}\right) \quad \text{and} \quad \theta_{234} = \theta_{234} + 180°, \tag{2.68}$$

and we can calculate S_{234} and C_{234}, which are used to calculate θ_3, as previously discussed.

Now, referring again to Equation (2.63), repeated here, we can calculate the sine and cosine of θ_2 as follows:

$$\begin{cases} p_xC_1 + p_yS_1 = C_{234}a_4 + C_{23}a_3 + C_2a_2, \\ p_z = S_{234}a_4 + S_{23}a_3 + S_2a_2. \end{cases}$$

Since $C_{12} = C_1C_2 - S_1S_2$ and $S_{12} = S_1C_2 + C_1S_2$, we get

$$\begin{cases} p_xC_1 + p_yS_1 - C_{234}a_4 = (C_2C_3 - S_2S_3)a_3 + C_2a_2, \\ p_z - S_{234}a_4 = (S_2C_3 + C_2S_3)\,a_3 + S_2a_2. \end{cases} \tag{2.69}$$

Treating this as a set of two equations in two unknowns and solving for C_2 and S_2, we get

$$\begin{cases} S_2 = \dfrac{(C_3a_3 + a_2)(p_z - S_{234}a_4) - S_3a_3(p_xC_1 + p_yS_1 - C_{234}a_4)}{(C_3a_3 + a_2)^2 + S_3^2a_3^2}, \\ C_2 = \dfrac{(C_3a_3 + a_2)(p_xC_1 + p_yS_1 - C_{234}a_4) + S_3a_3(p_z - S_{234}a_4)}{(C_3a_3 + a_2)^2 + S_3^2a_3^2}. \end{cases} \tag{2.70}$$

Although this is a large equation, all of its elements are known, and it can be evaluated. Thus,

$$\theta_2 = \tan^{-1}\frac{(C_3a_3 + a_2)(p_z - S_{234}a_4) - S_3a_3(p_xC_1 + p_yS_1 - C_{234}a_4)}{(C_3a_3 + a_2)(p_xC_1 + p_yS_1 - C_{234}a_4) + S_3a_3(p_z - S_{234}a_4)}. \tag{2.71}$$

Now that θ_2 and θ_3 are known, we can calculate

$$\theta_4 = \theta_{234} - \theta_2 - \theta_3. \tag{2.72}$$

Please remember that since there are two solutions for θ_{234} (Equation 2.68), there will be two solutions for θ_4 as well.

From 1,3 and 2,3 elements of Equation (2.67), we get

$$\begin{cases} S_5 = C_{234}(C_1a_x + S_1a_y) + S_{234}a_z, \\ C_5 = -C_1a_y + S_1a_x, \end{cases} \tag{2.73}$$

and

$$\theta_5 = \tan^{-1}\frac{(C_{234}(C_1a_x + S_1a_y) + S_{234}a_z}{S_1a_x - C_1a_y}. \tag{2.74}$$

$$\begin{bmatrix} C_5[C_{234}(C_1n_x + S_1n_y) + S_{234}n_z] & C_5[C_{234}(C_1o_x + S_1o_y) + S_{234}o_z] & 0 & 0 \\ -S_5(S_1n_x - C_1n_y) & -S_5(S_1o_x - C_1o_y) & & \\ -S_{234}(C_1n_x + S_1n_y) + C_{234}n_z & -S_{234}(C_1o_x + S_1o_y) + C_{234}o_z & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C_6 & -S_6 & 0 & 0 \\ S_6 & C_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.75}$$

From the 2,1 and 2,2 elements of Equation (2.75), we get

$$\theta_6 = \tan^{-1}\frac{-S_{234}(C_1n_x + S_1n_y) + C_{234}n_z}{-S_{234}(C_1o_x + S_1o_y) + C_{234}o_z}. \tag{2.76}$$

6

Final Solutions

$$ {}^{R}T_{H_{\text{DESIRED}}} = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}, $$

$$ \theta_1 = \tan^{-1}\left(\frac{p_y}{p_x}\right) \quad \text{and} \quad \theta_1 = \theta_1 + 180°, $$

$$ \theta_{234} = \tan^{-1}\left(\frac{a_z}{C_1 a_x + S_1 a_y}\right) \quad \text{and} \quad \theta_{234} = \theta_{234} + 180°, $$

$$ C_3 = \frac{(p_x C_1 + p_y S_1 - C_{234} a_4)^2 + (p_z - S_{234} a_4)^2 - a_2^2 - a_3^2}{2 a_2 a_3}, $$

$$ S_3 = \pm\sqrt{1 - C_3^2}, $$

$$ \theta_3 = \tan^{-1}\frac{S_3}{C_3}, $$

$$ \theta_2 = \tan^{-1}\frac{(C_3 a_3 + a_2)(p_z - S_{234} a_4) - S_3 a_3 (p_x C_1 + p_y S_1 - C_{234} a_4)}{(C_3 a_3 + a_2)(p_x C_1 + p_y S_1 - C_{234} a_4) + S_3 a_3 (p_z - S_{234} a_4)}, $$

$$ \theta_4 = \theta_{234} - \theta_2 - \theta_3, $$

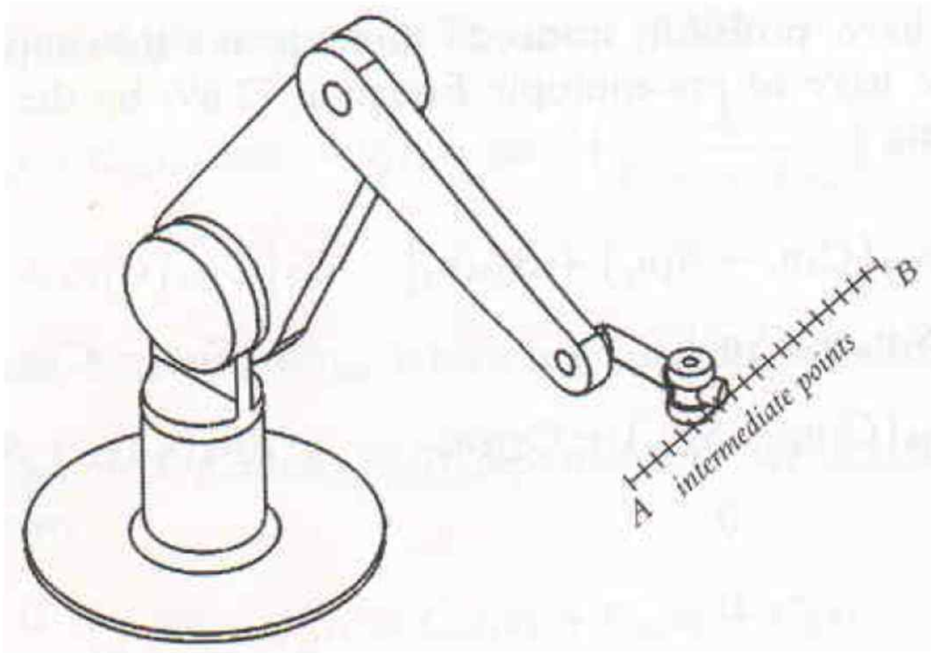
$$ \theta_5 = \tan^{-1}\frac{C_{234}(C_1 a_x + S_1 a_y) + S_{234} a_z}{S_1 a_x - C_1 a_y}, $$

$$ \theta_6 = \tan^{-1}\frac{-S_{234}(C_1 n_x + S_1 n_y) + C_{234} n_z}{-S_{234}(C_1 o_x + S_1 o_y) + C_{234} o_z}. $$

Inverse Kinematic Programming

- I.K 계산의 정확성 문제
  - 로봇의 accuracy 에 영향을 줌

- I.K 계산 시간 문제
  - 로봇이 지정된 경로 (직선, 원 등) 이동 시 빠른 계산 필요 (초당 50-200 회)
  - Intermediate points
  - 경로 이동 속도 및 편차에 영향을 줌



Degeneracy

- 정의
  – Inverse Kinematics 해가 무한 개 존재하는 위치/자세
  ➢ 무한 개의 configuration $(\theta_1, \cdots, \theta_n)$ 으로 도달 가능한 위치/자세
  ➢ 자유도 (DOF) 를 떨어뜨림 (de – generate)
  ➢ 제어기에서 위치 결정 불가능